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$SU(3)$ as a Group of Broken Symmetry in the Problems with Arbitrary Central Dynamics

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Abstract

An algebraic treatment of three-dimensional problems with arbitrary central field of force is proposed in terms of $SU(3)$.

1. *Introduction*

Description of physical problems in a group theoretical language means that a complete set of commuting physical observables is formed of group theoretical independent quantities. These quantities should be connected with dynamics of motion in a unique way.

In this article the class of three-dimensional problems with arbitrary central dynamics is considered (bounded motion). The description of the corresponding problems in the terms of the $O(4)$ group has been considered previously by Serebrennikov and Shabad (1973), Serebrennikov (1974), and Serebrennikov *et al.* (1975). The approach to the $SU(3)$ treatment is in fact the same. The description in terms of the $SU(3)$ group is of special interest, because $SU(3)$ is a symmetry group of the isotropic oscillator (Jauch and Hill, 1940; Demkov, 1953; Baker, 1956) and widely used in hadronic systematics.

We start the investigation in frames of the classical mechanics and shall deal with the Poisson brackets algebra of generators of infinitesimal canonical transformations.

We take the Casimir invariant[1] G of the $SU(3)$ group, the orbital momentum squared L^2 [which is the Casimir invariant of the $O(3)$ subgroup of the $SU(3)$] and the orbital momentum component l_3 for the above independent group

[1] In the problem under consideration the two Casimir invariants of the $SU(3)$ group are interrelated (see section 3). That is why we refer only to one of them.

quantities. They all commute among themselves and may be used as the complete set of independent integrals of motion. Then the Hamiltonian is the function of G, L^2 (the l_3 dependence is exluded by the central symmetry) (cf. Serebrennikov and Shabad, 1973).

The establishment of the connection with the dynamics of the motion consists in finding this function and the $SU(3)$ generators as explicit and unique functions of the position $\mathbf{r}$ and momentum $\mathbf{p}$. The transition from the Hamiltonian $H = \frac{1}{2}p^2 + V(r)$ to $H = H(G, L^2)$ has some advantages.

Firstly, the function H of the noncommuting variables $\mathbf{r}$ and $\mathbf{p}$ is replaced by one of the commuting group variables G, L^2. Secondly, the functions G and L^2 when substituted for by the operators have a standard set of eigenvalues.

The quantization that naturally follows from the given group theoretical treatment consists in replacing the group variables (and not $\mathbf{r}$ and $\mathbf{p}$) by operators.

The calculation of the spectrum is reduced to the set of their eigenvalues for the group invariants G and L^2. This quantization is quasiclassical by nature of its dynamical concerns.

The main difference between the $O(4)$ and the $SU(3)$ is that the group canonical transformations $O(4)$ function in such a way that the integral invariant

$$J = \oint \sum_{i=1}^{3} p_i dr_i$$

remains invariant. Thus the $O(4)$ group appears to be in this sense a group of dynamical symmetry of the arbitrary central problem. With respect to the Hamiltonian, however, the $O(4)$ is a group of broken symmetry.

In the Coulomb problem the Hamiltonian is expressed only by the integral invariants, and thus the $O(4)$ group is at the same time a symmetry group of the Hamiltonian. On the contrary, the group of canonical transformations $SU(3)$ does not leave the integral invariant unchanged. (Note that the combination $J + J_r$, where J is the integral invariant and J_r is the radial invariant, does remain unchanged.)

Thus $SU(3)$ is not a symmetry group in the general central problem in the above sense; $SU(3)$ is generally a group of broken symmetry with respect to the Hamiltonian. In the oscillatory problem $SU(3)$ is a symmetry group of the Hamiltonian, because $H = (\omega/\pi)(J + J_r)$ and thus H is a function of G. The dependence of $H = H(G, L^2)$ upon L^2 describes the way in which the symmetry is broken for a given potential and implies that not all of the generators of this group are integrals of the motion.

In section 2 it is shown that the generators of this group in the case of arbitrary central motion must be nonconserving. Exact expressions of these generators are given and their dynamical properties are described. The construction of the $SU(3)$ generators is traced in the Appendix.

In section 3 we build the Casimir invariants. The energy spectrum is obtained and the $SU(3)$-multiplet structure of the energy spectrum is described. The quantization rules obtained appear to be quasiclassical and equivalent to the

Bohr–Sommerfeld quantization rules and to the ones which followed from the $O(4)$ approach (Serebrennikov and Shabad, 1973).

2. *The Classical Nonconserving Generators of SU(3) and their Unique Connection with Dynamics of the Motion*

Let us show that not all of the generators of $SU(3)$ are integrals of the motion. All the generators Φ_α of the $SU(3)$ group, where $\alpha = 1, 2, 3, \ldots, 8$, commute with the Casimir function (commutativity in the Poisson brackets sense)

$$\{G, \Phi_\alpha\} = \frac{\partial G}{\partial L^2} \{L^2, \Phi_\alpha\} + \frac{\partial G}{\partial H} \{H, \Phi_\alpha\} = 0 \tag{2.1}$$

Three generators Φ_i, $i = 1, 2, 3$, as they are at the same time generators of the $SO(3)$ subgroup, commute with L^2. However, the other five, not connected with the angular momentum, do not commute with L^2. Thus, they do not commute with the Hamiltonian due to (2.1), i.e., they are not integrals of motion.

A degenerate case ought to be distinguished when $H(L^2, G)$ does not depend upon L. Then these generators are integrals of motion, as is clear from (2.1). It takes place in the oscillatory problem, where all five independent components of the symmetric tensor are conserved:

$$A_{ij} = r_i r_j + p_i p_j, \qquad \mathrm{Sp} A_{ij} = 2H \tag{2.2}$$

The $SU(3)$ symmetry of the Hamiltonian is broken in all other problems.

Let us write down the nonconserving generators (they will be constructed in the Appendix):

$$\Phi_4 = A_{12}, \Phi_5 = A_{13}, \Phi_6 = A_{23}, \Phi_7 = \tfrac{1}{2}(A_{11} - A_{22}), \Phi_8 = -\sqrt{(3/2)}(A_{33} - f/3) \tag{2.3}$$

where A_{ij}, $i, j = 1, 2, 3$, are components of the symmetric tensor.

$$A_{ij} = D_1 \frac{r_i r_j}{r^2} + D_2 \frac{(\mathbf{r} \times \mathbf{L})_i (\mathbf{r} \times \mathbf{L})_j}{r^2 L^2} + D_3 \frac{r_i (\mathbf{r} \times \mathbf{L})_j + r_j (\mathbf{r} \times \mathbf{L})_i}{r^2 L} \tag{2.4}$$

$$D_1 = \tfrac{1}{2} f \mp (\tfrac{1}{4} f^2 - L^2)^{1/2} \cos 2\xi \tag{2.5}$$

$$D_2 = \tfrac{1}{2} f \pm (\tfrac{1}{4} f^2 - L^2)^{1/2} \cos 2\xi \tag{2.6}$$

$$D_3 = \pm (\tfrac{1}{4} f^2 - L^2)^{1/2} \sin 2\xi \tag{2.7}$$

$$\xi = \int_{r_1(H, L^2)}^{r} \frac{1 - 2Fr^2}{r^2} \left(\frac{2H - 2V}{L^2} - \frac{1}{r^2} \right)^{-1/2} dr \tag{2.8}$$

$$F = -\frac{\partial f}{\partial L^2} \Big/ \frac{\partial f}{\partial H} \tag{2.9}$$

The function f is the trace of the tensor (2.4). It is connected with the Casimir function by means of the relation

$$f(H, L^2) = \sqrt{3G} = S_p A_{ij} \tag{2.10}$$

and is given by the expression

$$f(H, L^2) = 2(2J_r/\pi + \sqrt{L^2}) = (2/\pi)(J + J_r) \tag{2.11}$$

where J is the integral invariant $J = J_r + \pi\sqrt{L^2}$ and J_r is the radial invariant

$$J_r = \int_{r_{\min}}^{r_{\max}} [2(H - V)r^2 - L^2]^{1/2} \frac{dr}{r} \tag{2.12}$$

The lower integration limit in (2.9) is in the classically accessible domain $r_{\min} \leqslant r_1 \leqslant r_{\max}$. It fixes the initial direction of the principal axis of the tensor (2.4). A_{ij} with unknown $f = f(H)$, $F = 0$ becomes Fradkin's (1967) tensor.

Let us adduce the properties of the tensor A_{ij}, according to which it has been constructed:

(a) It is in the plane of the orbit:

$$A_{ij}L_i = A_{ij}L_j = 0 \tag{2.13}$$

(b) It commutes with its trace:

$$\{\mathrm{Sp}A_{ij}, H\} = 0 \tag{2.14}$$

(c) Its five independent components and three components of the angular momentum form the $SU(3)$ algebra (with respect to the Poisson brackets):

$$\{L_i, L_j\} = \epsilon_{ijk}L_k \tag{2.15}$$

$$\{A_{ij}, L_e\} = \epsilon_{ein}A_{nj} + \epsilon_{ejn}A_{ni} \tag{2.16}$$

$$\{A_{ij}, A_{em}\} = L_n(\delta_{ie}\epsilon_{njm} + \delta_{im}\epsilon_{nje} + \delta_{je}\epsilon_{nim} + \delta_{jm}\epsilon_{ien}) \tag{2.17}$$

where ϵ_{ijk} is the unit antisymmetric tensor; δ_{ik} is the Cronecker symbol.

(d) The eigenvectors $\mathbf{A}$ and $\mathbf{A} \times \mathbf{L}$ of the tensor A_{ij} [see (2.19)] are directed, respectively, to the perihelion and apohelion of the orbit, whenever the position vector $\mathbf{r}$ passes them. (We choose $r_1 = r_{\min}$ for definiteness).

(e) It is isotropic in the plane of motion for the circular trajectories.

Other properties of A_{ij} are as follows:

(f) In the oscillatory problem it turns into the known symmetric tensor which is an integral of motion. It is not difficult to verify this by direct substitution of the oscillatory potential into (2.4), if $r_1 = r_{\min}$.

(g) The direct calculation of the Poisson brackets for the A_{ij} tensor with the Hamiltonian gives

$$\{A_{ij}, H\} = 4FL(\tfrac{1}{4}f^2 - L^2)^{1/2}[A_i(\mathbf{A} \times \mathbf{L})_j + A_j(\mathbf{A} \times \mathbf{L})_i] \tag{2.18}$$

where

$$\mathbf{A} = \frac{\mathbf{r}}{r}\cos\xi + \frac{\mathbf{r}\times\mathbf{L}}{rL}\sin\xi \tag{2.19}$$

is the unit eigenvector of the tensor A_{ij}. Vector (2.19) differs from the "generalized Runge-Lenz vector" determined in our previous paper (Serebrennikov and Shabad, 1973) in values taken by F.

Tensor A_{ij}, when built according to (a), (b), and (c) alone, is not connected with the trajectory of the motion. It is determined by requirements (a), (b), and (c) in a nonunique way. The arbitrariness reduces to the choice of the functions $f(L^2, H) = \mathrm{Sp}A_{ij}$ and the constant of integration $r_1(L^2, H)$.

The requirements (d) and (e) determine the function $f(L^2, H)$ to be (2.11). (As for the phase r_1, we recall that we have fixed it as $r_1 = r_{\min}$ for convenience.)

Let us consider the properties (d) and (e) in detail and get (2.11).

To satisfy the condition (d) in the expression (2.8) is to demand that the coefficients D_1, D_2, D_3 (2.5)-(2.7) should take the same values each time the moving particle passes the perihelion ($r = r_{\min}$). The condition (d) leads to the fact that the tensor A_{ij} rests in the coordinate frame where the trajectory is closed. This frame rotates with the constant angular velocity F, according to (2.18) and (2.19). Taking into account the trigonometrical dependence of the D_1, D_2, D_3 functions of the variable r, the condition (d) is written in the following way:

$$2\xi(r_{\max}) - 2\xi(r_{\min}) = \pi \tag{2.20}$$

Let us consider the condition (e). There is no special direction in the plane of the orbit, when the latter is circular. That is why we demand the isotropy of the tensor A_{ij} in the plane of the orbit. To do this, let us decompose the tensor A_{ij} (2.4) into the products of its two orthonormal eigenvectors $\mathbf{A}$, $(\mathbf{A}\times\mathbf{L})/L$, lying in the plane of motion

$$A_{ij} = C_1 A_i A_j + C_2\frac{(\mathbf{A}\times\mathbf{L})_i(\mathbf{A}\times\mathbf{L})_j}{L^2} \tag{2.21}$$

where $\mathbf{A}$ is the vector (2.19). The eigenvalues C_1 and C_2 are given by the expressions

$$C_1 = \tfrac{1}{2}f \mp \sqrt{(\tfrac{1}{2}f)^2 - L^2} \tag{2.22}$$

$$C_2 = \tfrac{1}{2}f \pm \sqrt{(\tfrac{1}{2}f)^2 - L^2} \tag{2.23}$$

The third eigenvector is orthogonal to the plane of the motion and its corresponding eigenvalue is 0 and thus does not contribute to the decomposition (2.21). Isotropy of the tensor in the plane of motion implies the equality of the eigenvalues (2.22) and (2.23). Thus

$$\tfrac{1}{4}f^2\,|_{H=H(L^2)} = L^2 \tag{2.24}$$

The function H is given on the circular orbit parametrically

$$H = \tfrac{1}{2}r\frac{dV(r)}{dr} + V(r), \qquad L^2 = r^3\frac{dV(r)}{dr} \tag{2.25}$$

The condition (2.20) determines the function $F(L^2, H)$, when the limits of the integration $r_{\min}, r_{\max}$ are substituted into (2.8). The function $f(L^2, H)$ is found from the solution of the first-order partial differential equation (2.9). The boundary condition (2.24) follows from property (e). The expression (2.11) is the solution for arbitrary central motion. The dependence of the function $H(f, L^2)$ upon f, L^2 is given inexplicitly by

$$J_r(H, L^2) = \tfrac{1}{2}\pi(\tfrac{1}{2}f - \sqrt{L^2}), \qquad f = \sqrt{3G} \tag{2.26}$$

It should be noted that the conditions (d) and (e) could be obtained formally, but there is no logical necessity to do so.

3. *Casimir Operators and Quantization*

The Casimir operators $\hat{G}_2$ and $\hat{G}_3$ of the $SU(3)$ group are homogeneous polynomials of the generators of the second and the third order, respectively

$$\hat{G}_2 = \sum_{\alpha} \Phi_\alpha{}^2 \tag{3.1}$$

$$\hat{G}_3 = \sum_{\alpha,\beta,\gamma} d_{\alpha\beta\gamma}\Phi_\alpha\Phi_\beta\Phi_\gamma \tag{3.2}$$

where Φ_α are the standard generators of $SU(3)$ with the commutation relations

$$[\Phi_\alpha, \Phi_\beta] = if_{\alpha\beta\gamma}\Phi_\gamma \tag{3.3}$$

Here $d_{\alpha\beta\gamma}$ and $f_{\alpha\beta\gamma}$ are the well-known fully symmetrical and antisymmetrical structural constants, respectively. The eigenvalues of the operators (3.1) and (3.2) are the combinations of integers

$$G_2 = \tfrac{4}{3}(n^2 + q^2 + nq + 3n + 3q) \tag{3.4}$$

$$G_3 = \tfrac{4}{9}(n - q)(2n + q + 3)(2q + n + 3) \tag{3.5}$$

where $n, q = 0, 1, 2 \ldots$.

Let us calculate the classical Casimir functions. To do this the generators L_i, A_{ij} taken in the standard form (2.3) are substituted into (3.1) and (3.2). Then we get

$$G_2 = \tfrac{1}{3}f^2 \tag{3.6}$$

$$G_3 = \tfrac{1}{9}f^3 = \tfrac{1}{3}G_2 f \tag{3.7}$$

Thus, owing to the orthogonality of the angular momentum and A_{ij}, the Casimir functions G_2 and G_3 are the second and third power of the trace of the tensor. This implies the multiplication of the eigenvalues of the operators G_2 and f in G_3 (3.7). Therefore we have

$$q = 0, \qquad f = 2n + 3 \tag{3.8}$$

It is convenient to take the quantities f, L^2, L_3 as the complete set of the independent commuting integrals of motion.

Let us take operators for the generators of $SU(3)$ instead of the classical expressions, as has been done for $O(4)$ by Serebrennikov and Shabad (1973). For the Hamiltonian H we postulate the same functional dependence on the operators $\hat{f}$, $\hat{L}^2$, as in the classical problem (2.26). We bear in mind that the operators $\hat{f}$ and $\hat{L}^2$ commute $[\hat{f}, \hat{L}^2] = 0$ and thus no question arises concerning the ordering of them within the function $\hat{H}(\hat{f}, \hat{L}^2)$. Note that the resulting quantum dynamics is not, generally, equivalent to the usual one, when one takes the classical functional dependence $H(r, p) = \frac{1}{2}p^2 + V(r)$ on the position r and momentum p for the quantum Hamiltonian function of the operators $\hat{r}$ and $\hat{p}$. The two quantum dynamics coincide only with quasiclassical accuracy.

Let us take the states in which L^2, L_3 and f have definite values. The eigenvalues of L_3, L^2, f are

$$\hat{L}_3\Psi_{nlm} = hm\Psi_{nlm}, \qquad -l \leqslant m \leqslant l \tag{3.9}$$

$$\hat{L}^2\Psi_{nlm} = h^2 l(l+1), \qquad l = \begin{cases} 0, 2, 4, \ldots, n \\ 1, 3, 5, \ldots, n \end{cases} \tag{3.10}$$

$$\hat{f}\Psi_{nlm} = h(2n+3)\Psi_{nlm}, \qquad n = 0, 1, 2, \ldots \tag{3.11}$$

where (3.11) comes from (3.8). Then the eigenvalues of the Hamiltonian H will be the following:

$$\hat{H}(\hat{L}^2, \hat{f})\Psi_{nlm} = H(h^2 l(l+1), h(2n+3))\Psi_{nlm} \tag{3.12}$$

since the Hamiltonian $\hat{H}$ is given in terms of L^2 and f, and has a common system of eigenfunctions with them. These relations contain the quantization rules

$$J_r(E_{nl}, h^2 l(l+1)) = \tfrac{1}{2}\pi h\left[n + \tfrac{3}{2} - \sqrt{l(l+1)}\right] \tag{3.13}$$

The integer n labels the $SU(3)$ multiplet corresponding to the $D(n, 0)$ representation with the dimension $\frac{1}{2}(n+1)(n+2)$. The functions Ψ_{nlm} (n is fixed) are its basis vectors. The dependence $E_{nl}(l)$ upon l (n = const) removes the degeneration within the multiplet, i.e., the energy levels split in the general case of the arbitrary central problem. The quantum number n is connected with the radial quantum number n_r in the following way:

$$n = 2n_r + l \tag{3.14}$$

These quantization rules are equivalent in fact to the Bohr-Sommerfeld quantization rules and to the ones obtained within the $O(4)$ approach.

If we substitute the quantum number (3.14) into (3.13) and make the known quasiclassical replacement $l(l+1) \to (l+\frac{1}{2})^2$ being expedient in some cases, we shall get the known expression for spectrum

$$J_r(E_{nl}, l(l+1)) = \pi h(n_r + \tfrac{1}{2})$$

The energy spectrum following from the $O(4)$ treatment may be obtained if we put the "hydrogenous" n_{hyd} quantum number into (3.13) connected with the "oscillatory" one n_{osc} by relation $n_{\text{osc}} = 2n_{\text{hyd}} - l - 2$.

4. *Concluding Remarks*

The $SU(3)$ group acts in the phase space of the arbitrary central problem, the symmetry of the Hamiltonian proving to be intrinsically broken. This is the essence of the approach. The generators of the $SU(3)$ group are connected with the dynamics of the motion in a simple way. The set of these generators is sufficient to fix the trajectory of the arbitrary bounded motion. It allows us to perform the whole algebraic description of the motion. The generalization of the results for n-dimensional and relativistic cases is not difficult, and the $O(4)$ treatment of relativistic problems without spin is given by one of the present authors (Serebrennikov, 1974). The problem with noncentral potential may be described within the same group-theoretical scheme.

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Appendix: Construction of the Nonconserving Generators of the SU(3) Group

Let us introduce a symmetric tensor A_{ij} in the most general form, taking into account (2.13):

$$A_{ij} = B_1 r_i r_j + B_2 p_i p_j + B_3 (r_i p_j + r_j p_i) \tag{A1}$$

$$\mathrm{Sp} A_{ij} = B_1 r^2 + B_2 p^2 + 2B_3 (\mathbf{rp}) = f(L^2, H) \tag{A2}$$

where B_1, B_2, B_3 are arbitrary functions of r, L^2, H. To find the coefficients B_1, B_2, B_3 is to find the tensor A_{ij}.

Let us require that the Poisson brackets of the tensor A_{ij} with its trace $\mathrm{Sp} A_{ij} = f(L^2, H)$ disappear:

$$\{A_{ij}, \mathrm{Sp} A_{ij}\} = 0 \tag{A3}$$

Equating the scalar coefficients at the independent tensors with zero, we have

$$\begin{gathered} 4F[B_1(\mathbf{rp}) + B_3 p^2] - 2B_3 \frac{dV}{dr}\frac{1}{r} + \frac{\partial B_1}{\partial r}\frac{(\mathbf{rp})}{r} = 0 \\ 4F[B_2(\mathbf{rp}) + B_3 r^2] - 2B_3 - \frac{\partial B_2}{\partial r}\frac{(\mathbf{rp})}{r} = 0 \\ 2F[B_2 p^2 - B_1 r^2] - B_2 \frac{dV}{dr}\frac{1}{r} + B_1 + \frac{\partial B_1}{\partial r}\frac{(\mathbf{rp})}{r} = 0 \end{gathered} \tag{A4}$$

where

$$F = -\frac{\partial f}{\partial L^2} \Big/ \frac{\partial f}{\partial H} \tag{A5}$$

Let us express the function B_1 in terms of B_2 and B_3 according to (A2)

$$B_1 = (1/r^2)[f - B_2 p^2 - 2B_3(\mathbf{rp})] \tag{A6}$$

and make the substitution

$$B_2 = r^2 D_2 / L^2 \tag{A7}$$

Then the set (A4) may be transformed to the second-order inhomogenous equation with respect to D_2:

$$a \frac{\partial^2 D_2}{\partial r^2} + \frac{1}{2} \frac{\partial D_2}{\partial r} \frac{da}{dr} + D_2 = \frac{f}{2} \tag{A8}$$

where

$$a = \frac{(\mathbf{rp})^2 r^2}{4L^2(1 - 2Fr^2)^2}$$

A particular solution of equation (A8) without the right-hand side is the function

$$A(L^2, H) \cos 2\xi \tag{A9}$$

where

$$\xi(r) = \frac{1}{2} \int_{r_1}^{r} \frac{dr}{\sqrt{a}}$$

$A(L^2, H)$ and $r_1(L^2, H)$ are constants with respect to r. The choice of the constant r_1 is the choice of the origin of the azimuth angle.

The function

$$D_2 = \tfrac{1}{2}f + (\tfrac{1}{2}f + A) \cos 2\xi \tag{A10}$$

is the general solution of (A8). The function $A(L^2, H)$ in the expression (A10) is arbitrary function of L^2, H. We may go from the functions B_1, B_2, B_3 to the functions D_1, D_2, D_3.

$$\begin{aligned} B_1 &= \frac{1}{r^2}\left[D_1 + D_2 \frac{(\mathbf{rp})^2}{L^2} + 2D_3 \frac{(\mathbf{rp})}{L}\right] \\ B_2 &= \frac{r^2}{L^2} D_2 \\ B_3 &= -\left[D_2 \frac{\mathbf{rp}}{L^2} + D_3 \frac{1}{\sqrt{L^2}}\right] \end{aligned} \tag{A11}$$

then

$$\begin{aligned} D_1 &= \tfrac{1}{2}f - (\tfrac{1}{2}f + A) \cos 2\xi \\ D_2 &= \tfrac{1}{2}f + (\tfrac{1}{2}f + A) \cos 2\xi \\ D_3 &= -(\tfrac{1}{2}f + A) \sin 2\xi \end{aligned} \tag{A12}$$

If one goes from the directions **r**, **p** to the mutually orthogonal ones **r**, **r** x **L** in the tensor (A1), then the functions D_1, D_2, D_3 are the coefficients by the unit tensors

$$A_{ij} = D_1 \frac{r_i r_j}{r^2} + D_2 \frac{(r \times L)_i (r \times L)_j}{r^2 L^2} + D_3 \frac{r_i (r \times L)_j + r_j (r \times L)_i}{r^2 \sqrt{L^2}}$$

$$\mathrm{Sp} A_{ij} = D_1 + D_2 \tag{A13}$$

This form of the tensor is more convenient. We determine the function $A(L^2, H)$ from requirements (2.16) and (2.17). We give the function A without detailed computation:

$$A + \tfrac{1}{2}f = \pm\sqrt{\tfrac{1}{4}f^2 - L^2} \tag{A14}$$

Then

$$\begin{aligned} D_1 &= \tfrac{1}{2}f \mp (\tfrac{1}{4}f^2 - L^2)^{1/2} \cos 2\xi \\ D_2 &= \tfrac{1}{2}f \pm (\tfrac{1}{4}f^2 - L^2)^{1/2} \cos 2\xi \\ D_3 &= \mp (\tfrac{1}{4}f^2 - L^2)^2 \sin 2\xi \end{aligned} \tag{A15}$$

The procedure for uniquely determining function $f(L^2, H)$ is given in section 3.

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